

INVARIANTS OF UPPER MOTIVES

OLIVIER HAUTION

ABSTRACT. Let H be a homology theory for algebraic varieties over a field k . To a complete k -variety X , one naturally associates an ideal $H_X(k)$ of the coefficient ring $H(k)$. We show that, when X is regular, this ideal depends only on the upper Chow motive of X . This generalises the classical results asserting that this ideal is a birational invariant of smooth varieties for particular choices of H , such as the Chow group. When H is the Grothendieck group of coherent sheaves, we obtain a lower bound on the canonical dimension of varieties. When H is the algebraic cobordism, we give a new proof of a theorem of Levine and Morel. Finally we discuss some splitting properties of geometrically unirational field extensions of small transcendence degree.

1. INTRODUCTION

The canonical dimension of a smooth complete algebraic variety measures to which extent it can be rationally compressed. In order to compute it, one often rather study the p -local version of this notion, called canonical p -dimension (p is a prime number). In this paper, we consider the relation of p -equivalence between complete varieties, constructed so that p -equivalent varieties have the same canonical p -dimension. In particular, two complete varieties X and Y are p -equivalent, for any p , as soon as there are rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$. In order to obtain restrictions on the possible values of the canonical p -dimension of a variety, one is naturally led to study invariants of p -equivalence. We give a systematic way to produce such invariants (and in particular, birational invariants), starting from a homology theory. We provide examples related to K -theory and cycle modules. We then describe the relation between two such invariants of a complete variety X : its index n_X , and the integer d_X defined as the g.c.d. of the Euler characteristics of the coherent sheaves of \mathcal{O}_X -modules. The latter invariant contains both arithmetic and geometric informations; this can be used to give

Date: October 30, 2012.

2010 Mathematics Subject Classification. 14C25.

Key words and phrases. upper motives, canonical dimension, Grothendieck group, algebraic cobordism.

This work was financed by the EPSRC Responsive Mode grant EP/G032556/1.

bounds on the possible values of the index n_X (an arithmetic invariant) in terms of the geometry of X . For instance, a smooth, complete, geometrically rational (or merely geometrically rationally connected, when k has characteristic zero) variety of dimension $< p-1$ always has a closed point of degree prime to p . Some consequences of this statement are given in Corollary 7.8.

A result of Zainoulline [Zai10] states that, for a smooth complete variety X over a field of characteristic not p , we have

$$\dim X \geq (p-1)(v_p(\chi(X, \mathcal{O}_X)) - v_p(n_X)).$$

and moreover that X is p -incompressible in case of equality. Here we improve this result, and obtain the following bound on the canonical p -dimension of a regular complete variety X over a field characteristic not p (and also a weaker statement in characteristic p)

$$\mathrm{cdim}_p X \geq (p-1)(v_p(d_X) - v_p(n_X)).$$

We always obtain some bound on the canonical p -dimension, even when it is not equal to the dimension. By contrast, the result of [Zai10], or more generally any approach based on the degree formula (see [Mer03]), does not directly say anything about the canonical dimension of varieties which are not p -incompressible. One can sometimes circumvent this problem by exhibiting a smooth complete variety Y of dimension $\mathrm{cdim}_p(X)$ which is p -equivalent to X , and use the degree formula to prove that Y is p -incompressible (see [Mer03, §7.3] for the case of quadrics). But this requires to explicitly produce such a variety Y , and to find an appropriate characteristic number for Y .

Let us mention that the bound above tends to be sharp only when $\mathrm{cdim}_p(X)$ is not too large, compared with p (see Example 6.7).

Another aspect of the technique presented here concerns its application to the algebraic cobordism Ω of Levine and Morel. Their construction uses two distinct results known in characteristic zero:

- (a) the resolution of singularities [Hir64], and
- (b) the weak factorisation theorem [AKMW02, Wlo03].

Some care is taken in the book [LM07] to keep track of which result uses merely (a), or the combination of (a) and (b); most deeper results actually use both. In [Hau12b], we proved that the existence of a weak form of Steenrod operations (which may be considered as a consequence of the existence of algebraic cobordism) only uses (a). Moreover, we proved that it suffices, for the purpose of the construction of these operations, to have a p -local version of resolution of singularities, which has been recently obtained in characteristic different from p by Gabber. By contrast, it is not clear what

would be a p -local version of the weak factorisation theorem. In the present paper, we extend the list of results using only (a) by proving the theorem below, whose original proof in [LM07, Theorem 4.4.17] was based on (b) (and (a)). Our approach moreover allows us to give a p -local version of this statement, in terms of p -equivalence.

Theorem. *Assume that the base field k admits resolution of singularities. The ideal $M(X)$ of $\Omega(k)$ generated by classes of smooth projective varieties Y , of dimension $< \dim X$, and admitting a morphism $Y \rightarrow X$, is a birational invariant of a smooth projective variety X .*

2. p -EQUIVALENCE

We denote by k a fixed base field. We will also denote by k its spectrum. A variety will be an integral, separated, finite type scheme over k . The function field of a variety X (and its spectrum) will be denoted by $k(X)$. When K/k is a field extension, we write X_K for $X \times_k \text{Spec } K$. The letter p will always denote a prime number.

A prime correspondence, or simply a *correspondence*, $Y \rightsquigarrow X$ is a diagram of varieties and proper morphisms $Y \leftarrow Z \rightarrow X$, where the map $Y \rightarrow Z$ is generically finite. The degree of this map is the *multiplicity* of the correspondence.

Following [KM, §3], we say that two varieties X and Y are *p -equivalent* if there are correspondences $Y \rightsquigarrow X$ and $X \rightsquigarrow Y$ of multiplicity prime to p . It is equivalent to require that each of the schemes $X_{k(Y)}$ and $Y_{k(X)}$ have a closed point of degree prime to p .

Note that a rational map $Y \dashrightarrow X$ between complete varieties gives rise to a correspondence $Y \rightsquigarrow X$ of multiplicity 1. Thus two complete varieties X and Y are p -equivalent, for any p , as soon as there are rational maps $Y \dashrightarrow X$ and $X \dashrightarrow Y$.

We will in general restrict our attention to complete regular varieties. In this case the relation of p -equivalence becomes transitive. A consequence of a theorem of Gabber [ILO, X, Theorem 2.1] is that, when the characteristic of k is not p , any complete variety is p -equivalent to a complete regular variety (of the same dimension).

Let X be a complete regular variety. Its canonical p -dimension $\text{cdim}_p(X)$ can be defined as the least dimension of an integral closed subvariety $Z \subset X$ admitting a correspondence $X \rightsquigarrow Z$ of multiplicity prime to p [KM06, Corollary 4.12].

It is proven in [KM, Lemma 3.6] that smooth p -equivalent varieties have the same canonical p -dimension. A slight modification of the arguments used there, together with Gabber's theorem, yields the following statement.

Proposition 2.1. *Assume that the characteristic of k is different from p . Let X be a complete regular variety. Then $\mathrm{cdim}_p(X)$ is the least dimension of a complete regular variety p -equivalent to X .*

Remark 2.2 (Upper motives). Let X be a complete smooth variety. A summand of the Chow motive of X with \mathbb{F}_p -coefficients is called *upper* if it is defined by a projector of multiplicity 1 $\in \mathbb{F}_p$ [Kar, Definition 2.10].

If two smooth complete varieties have a common upper summand with \mathbb{F}_p -coefficients, then they are p -equivalent. The converse is true if the varieties are geometrically split and satisfy Rost nilpotence (see [Kar, Corollary 2.15]).

3. RING THEORIES AND MODULES

We denote by \mathbf{Ab} the category of abelian groups. Let \mathcal{V} be a full subcategory of the category of varieties and proper morphisms. We also fix a full subcategory \mathbf{Reg} of \mathcal{V} (below \mathbf{Reg} will be the subcategory consisting of either regular varieties, or smooth varieties). Let R and H be two functors $\mathcal{V} \rightarrow \mathbf{Ab}$. We consider some conditions on R and H whose purpose is to make the proof of Lemma 4.1 below work.

Conditions 3.1. Let $X \in \mathcal{V}$, $S \in \mathbf{Reg}$, and $f: X \rightarrow S$ be a proper, dominant, generically finite morphism of degree d .

- (R1) The group $R(S)$ has a structure of an associative ring with unit 1_S .
- (R2) It is possible to find an element $u \in R(X)$ such that the element $f_*u - d \cdot 1_S$ is nilpotent in the ring $R(S)$.
- (H1) There is a morphism of abelian groups

$$R(X) \otimes H(S) \rightarrow H(X) \quad ; \quad x \otimes s \mapsto x \cdot_f s.$$

- (H2) The group $H(S)$ has a structure of an $R(S)$ -module such that

$$f_*(x \cdot_f s) = (f_*x) \cdot s.$$

We now describe some classical examples of such functors R and H .

3.1. Quillen K -theory. (See [Qui73, §7]) Let \mathcal{V} be the category of varieties and proper morphisms, \mathbf{Reg} the full subcategory of regular varieties. Let $X \in \mathcal{V}$. We let $R(X)$ be the Grothendieck group $K'_0(X)$ of the category of coherent sheaves of \mathcal{O}_X -modules, and $H(X)$ be the m -th K -group $K'_m(X)$ of this category.

We denote by $K_m(X)$ the m -th K -group of the category of locally free coherent sheaves on X . The tensor product induces a morphism

$$(1) \quad K_m(X) \otimes K'_0(X) \rightarrow K'_m(X) \quad ; \quad x \otimes y \mapsto x \cap y.$$

When S is regular, the map $-\cap[\mathcal{O}_S]$ induces an isomorphism

$$(2) \quad \varphi_S: K_m(S) \rightarrow K'_m(S).$$

With $m = 0$, the combination of (2) and (1) gives (R1).

When $f: X \rightarrow S$ is a proper morphism, we have a morphism

$$K'_0(X) \otimes K'_m(S) \rightarrow K'_m(X) \quad ; \quad x \otimes s \mapsto x \cdot_f s = f^* \circ \varphi_S^{-1}(s) \cap x,$$

proving (H1). Then (H2) follows from the projection formula.

We prove (R2) for $u = [\mathcal{O}_X]$. The element $x = f_*u - d \cdot [\mathcal{O}_S]$ belongs to the kernel of the restriction to the generic point morphism $K'_0(S) \rightarrow K'_0(k(S))$ (see e.g. [Hau, Lemma 2.4]). This amounts to saying that its unique antecedent $y \in K_0(S)$ under (2) (with $m = 0$) has rank zero. Thus for any n , its n -th power y^n belongs to the n -th term of the gamma filtration. The image by (2) (with $m = 0$) of this term is contained in the n -th term topological filtration [sga71, Exposé X, Corollaire 1.3.3]. The latter vanishes when $n > \dim S$, hence so does x^n .

3.2. Chow groups and cycle modules. Let us sketch how Chow groups can be made to fit into this framework, although the situation is somewhat degenerate. Let \mathcal{V} be the category of varieties and proper morphisms, and R be the Chow group CH. The property (R2) is verified with $u = [X]$, since $f_*[X] = d \cdot [S]$.

When **Reg** consists of smooth varieties, (R1) is classical. Let M be a cycle module [Ros96, Definition 2.1]. One checks again that the conditions (H1) and (H2) are satisfied for H the Chow group $A_*(-; M)$ with coefficients in M [Ros96, p.356].

When **Reg** consists of all regular varieties, and M is Quillen K -theory, the conditions (R1), (H1) and (H2) are verified in [Gil81, §8] using Bloch's formula.

3.3. Algebraic cobordism. Assume that k admits resolution of singularities. Let \mathcal{V} be the category of quasi-projective varieties and projective morphisms, and **Reg** the full subcategory of smooth quasi-projective varieties. We take for $R = H$ the algebraic cobordism Ω of [LM07]. We prove properties (R1), (H1), (H2), and, under the additional assumption that f is separable, we prove (R2).

Property (R1) is a classical property of Ω . We have by [LM07, Lemma 2.4.15]

$$\Omega(X) = \operatorname{colim}_{\mathcal{C}} \Omega(Y),$$

where the colimit is taken over the category \mathcal{C} of projective X -schemes Y which are smooth k -varieties; if $g: Y \rightarrow Z$ is a morphism in \mathcal{C} , the transition map is the push-forward $g_*: \Omega(Y) \rightarrow \Omega(Z)$. Then for $Y \in \mathcal{C}$, we can make $\Omega(S)$ act on $\Omega(Y)$ using the pull-back along $Y \rightarrow S$ and the ring structure on $\Omega(Y)$. The map g_* is easily seen to be $\Omega(S)$ -linear using the projection formula. This gives an action on the colimit

$$(3) \quad \Omega(X) \otimes \Omega(S) \rightarrow \Omega(X),$$

proving (H1). Then property (H2) follows formally from the projection formula in the smooth case.

For any $T \in \mathcal{V}$, we consider the subgroup $\Omega(T)^{(n)}$ of $\Omega(T)$ generated by the images of g_* , where g runs over the projective morphisms $W \rightarrow T$ whose image has codimension $\geq n$ in T . When $T = S$ is smooth, one checks, using reduction to the diagonal and the moving lemma [LM07, Proposition 3.3.1], that the subgroups $\{\Omega(S)^{(n)}, n \geq 0\}$ define a ring filtration on $\Omega(S)$. Since $\Omega(S)^{(\dim X+1)} = 0$, any element of $\Omega(S)^{(1)}$ is nilpotent. Let u be the class in $\Omega(X)$ of any resolution of singularities of X . Since f separable, the element $f_*u - d \cdot 1_S$ vanishes when restricted to some non-empty open subscheme of S by [LM07, Lemma 4.4.5]. By the localisation sequence [LM07, Theorem 3.2.7], this means that this element belongs to $\Omega(S)^{(1)}$, proving (R2).

Let us mention that the pair $(R, H) = (\Omega(-), \Omega(-)_{(m)})$ satisfies Conditions 3.1, with f separable in (R2) (where $\Omega(T)_{(m)} = \Omega(T)^{(\dim T - m)}$). Indeed the main point is to see that the map (3) descends to a map

$$\Omega(X) \otimes \Omega^{(n)}(S) \rightarrow \Omega^{(n)}(X).$$

This can be seen using the fact that pull-backs along morphisms of smooth varieties respect the filtration by codimension of supports, a consequence of the moving lemma mentioned above. When $n = 1$, this can also be proved directly, using the fact that f is dominant and the localisation sequence.

4. THE SUBGROUP $H_X(k)$

In this section, (R, H) will be a pair of functors $\mathcal{V} \rightarrow \mathbf{Ab}$ satisfying Conditions 3.1. We assume that the base k belongs to \mathcal{V} . If $X \in \mathcal{V}$ is complete, its structural morphism $x: X \rightarrow k$ is then in \mathcal{V} . We consider the subgroup

$$H_X(k) = \text{im}(x_*: H(X) \rightarrow H(k)) \subset H(k).$$

Lemma 4.1. *Let $S \in \text{Reg}$ and $X \in \mathcal{V}$. Let $f: X \rightarrow S$ be a proper, dominant, generically finite morphism of degree d . Then the map*

$$f_*: H(X) \otimes \mathbb{Z}[1/d] \rightarrow H(S) \otimes \mathbb{Z}[1/d]$$

is surjective.

Proof. Using (R2), choose $u \in R(X)$ such that $f_*u - d \cdot 1_S$ is nilpotent. The element f_*u is then invertible in the ring $R(S) \otimes \mathbb{Z}[1/d]$. The lemma follows, since we have by (H2), for any $s \in H(S)$,

$$f_* (u \cdot_f ((f_*u)^{-1} \cdot s)) = s. \quad \square$$

Proposition 4.2. *Let X, Y be complete varieties, with $X \in \mathcal{V}$ and $Y \in \text{Reg}$. Let $Y \rightsquigarrow X$ be a correspondence of multiplicity prime to p . Then, as subgroups of $H(k) \otimes \mathbb{Z}_{(p)}$, we have*

$$H_Y(k) \otimes \mathbb{Z}_{(p)} \subset H_X(k) \otimes \mathbb{Z}_{(p)}.$$

Proof. Let $Y \leftarrow Z \rightarrow X$ be a diagram giving the correspondence. By Lemma 4.1, we have

$$H_Y(k) \otimes \mathbb{Z}_{(p)} \subset H_Z(k) \otimes \mathbb{Z}_{(p)}.$$

Since there is a proper morphism $Z \rightarrow X$, we have

$$H_Z(k) \subset H_X(k). \quad \square$$

Corollary 4.3. *Assume that two complete varieties $Y, X \in \text{Reg}$ are p -equivalent. Then, as subgroups of $H(k) \otimes \mathbb{Z}_{(p)}$, we have*

$$H_Y(k) \otimes \mathbb{Z}_{(p)} = H_X(k) \otimes \mathbb{Z}_{(p)}.$$

Corollary 4.4. *Let $Y, X \in \text{Reg}$ be two complete varieties. Assume that there are rational maps $Y \dashrightarrow X$ and $X \dashrightarrow Y$. Then $H_Y(k) = H_X(k)$.*

We now come back to the examples of theories given in section 3.

4.1. Chow groups. We have $\text{CH}(k) = \mathbb{Z}$, and for a complete variety X ,

$$\text{CH}_X(k) = n_X \mathbb{Z},$$

where n_X is the index of X , defined as the g.c.d. of the degrees of closed points of X . We denote the p -adic valuation of n_X by

$$n_p(X) = v_p(n_X).$$

We will need the following slightly more precise version of Proposition 4.2, when $R = H = \text{CH}$.

Proposition 4.5. *Let X and Y be complete varieties, with Y regular. Let $Y \rightsquigarrow X$ be a correspondence of multiplicity m . Then*

$$n_X \mid m \cdot n_Y.$$

Proof. Let $Y \xleftarrow{f} Z \xrightarrow{g} X$ be a diagram giving the correspondence. As explained in 3.2, we have a pull-back $f^*: \mathrm{CH}(Y) \rightarrow \mathrm{CH}(Z)$ (corresponding in the notations of (H1) to the association $y \mapsto [Z] \cdot_f y$). Let $y \in \mathrm{CH}_0(Y)$ be such that $\deg y = n_Y$. Then we have

$$n_X \mid \deg \circ g_* \circ f^* y = \deg \circ f_* \circ f^* y = \deg((f_*[Z]) \cdot y) = \deg(m \cdot y) = m \cdot n_Y. \quad \square$$

4.2. Cycle modules. When M be a cycle module, we have $A_*(k; M) = M(k)$. For a smooth (or merely regular when M is Quillen K -theory) complete variety X , consider the subgroup of $M(k) \otimes \mathbb{Z}_{(p)}$

$$\bigcup_L \mathrm{im} (M(L) \rightarrow M(k)) \otimes \mathbb{Z}_{(p)},$$

where L runs over the finite field extensions of k such that $X(L) \neq \emptyset$. Then Corollary 4.3 asserts that the subgroup above only depends on the p -equivalence class of X .

Remark 4.6. The group $A_0(X; M)$ itself is known to be a birational invariant of a complete smooth variety X (see [Ros96, Corollary 12.10] and [KM, Appendix RC]).

4.3. Grothendieck group. We have $K'_0(k) = \mathbb{Z}$, and for a complete variety X ,

$$(K'_0)_X(k) = d_X \mathbb{Z},$$

for a uniquely determined positive integer d_X . The integer d_X is the g.c.d. of the integers

$$\chi(X, \mathcal{G}) = \sum_i (-1)^i \dim_k H^i(X, \mathcal{G}),$$

where \mathcal{G} runs over the coherent sheaves of \mathcal{O}_X -modules. We denote the p -adic valuation of d_X by

$$d_p(X) = v_p(d_X).$$

Let us record for later reference a consequence of Proposition 4.2.

Proposition 4.7. *Let X and Y be complete varieties, with Y regular. Let $Y \rightsquigarrow X$ be a correspondence of multiplicity prime to p . Then*

$$d_p(X) \leq d_p(Y).$$

4.4. Algebraic cobordism. We say that two varieties X and Y are *separably p -equivalent* if there are diagrams of projective morphisms $Y \xleftarrow{f} Z \rightarrow X$ and $X \xleftarrow{g} Z' \rightarrow Y$ such that f and g are both separable and generically finite of degrees prime to p . For a projective variety X , and an integer $n \geq 0$, we write $\Omega_X(k)^{(n)}$ for the subgroup of $\Omega(k)$ generated by classes of smooth projective varieties Y admitting a morphism $Y \rightarrow X$ with image of codimension $\geq n$. In particular $\Omega_X(k)^{(0)} = \Omega_X(k)$. Corollary 4.3 gives:

Proposition 4.8. *Assume that k admits resolution of singularities. Let Y and X be smooth projective varieties of the same dimension, which are separably p -equivalent. Then, as subgroups of $\Omega(k) \otimes \mathbb{Z}_{(p)}$, for $n \geq 0$,*

$$\Omega_X(k)^{(n)} \otimes \mathbb{Z}_{(p)} = \Omega_Y(k)^{(n)} \otimes \mathbb{Z}_{(p)}.$$

For a projective variety X , let $M(X) \subset \Omega(k)$ be the ideal generated by classes of smooth projective varieties Y of dimension $< \dim X$ admitting a morphism $Y \rightarrow X$. We have $M(X) \subset \Omega_X(k)^{(1)}$; when k admits weak factorisation, this inclusion is an equality [LM07, Theorem 4.4.16].

The subgroup generated by classes of smooth projective varieties of dimension $< \dim X$ is a direct summand of $\Omega(k)$. The projection of $\Omega_X(k)^{(1)}$ to this summand generates the ideal $M(X)$. In particular we obtain the following statement, which was proved in [LM07, Theorem 4.4.17, 1.] under the additional assumption that k admits weak factorisation.

Proposition 4.9. *Assume that k admits resolution of singularities. Then the ideal $M(X)$ of $\Omega(k)$ is a birational invariant of a smooth projective variety X .*

5. RELATIONS BETWEEN n_X AND d_X

We have the obvious relation $d_X \mid n_X$. When $\dim X = 0$, we have $n_X = d_X$. A consequence of the next theorem is that $d_p(X) = n_p(X)$ when $\dim X < p - 1$.

Theorem 5.1. *Let X be a complete variety. Assume that one of the following conditions holds.*

- (i) *The characteristic of k is not p .*
- (ii) *We have $\dim X < p(p - 1)$.*
- (iii) *The variety X is regular.*

Then

$$n_p(X) \leq d_p(X) + \left\lceil \frac{\dim X}{p - 1} \right\rceil.$$

Proof. First note that we may assume that X is projective over k by Chow's lemma. Indeed if $X' \rightarrow X$ is an envelope [Ful98, Definition 18.3], then it follows from [Ful98, Lemma 18.3] that $n_X = n_{X'}$ and $d_X = d_{X'}$.

Note also that the group $K'_0(X)$ is generated by the classes $[\mathcal{O}_Z]$, where Z runs over the closed subvarieties of X . Therefore

$$d_X = \gcd\{\chi(Z, \mathcal{O}_Z); Z \subset X\}.$$

Then under the assumption (ii), the result follows from [Hau12a, Proposition 9.1]. Under the assumption (i), it follows from a similar argument based on [Hau12a, Theorem 4.2].

Now we assume (iii), and sketch a proof of the statement. We identify the groups $K'_0(X)$ and $K_0(X)$ using (2), and take $a \in K_0(X)$. We apply the Grothendieck-Riemann-Roch theorem [Ful98, Theorem 18.2] to the structural morphism $x: X \rightarrow k$, and get in $\mathbb{Q} = K'_0(k) \otimes \mathbb{Q} = \text{CH}(k) \otimes \mathbb{Q}$

$$(4) \quad x_* a = \text{ch} \circ x_* a = x_* \circ \text{Td}(T_x) \circ \text{ch} a.$$

Here $\text{ch}: K_0(X) \rightarrow \text{CH}(X) \otimes \mathbb{Q}$ is the Chern character, Td the Todd class, $T_x \in K_0(X)$ the virtual tangent bundle of the local complete intersection morphism x . One sees, using the splitting principle, that, for all n

$$p^{[n/(p-1)]} \cdot \text{ch}^n a \in \text{im} \left(\text{CH}^n(X) \otimes \mathbb{Z}_{(p)} \rightarrow \text{CH}^n(X) \otimes \mathbb{Q} \right).$$

A similar statement for the n -th component of the Todd class is proved in [Hau12a, Lemma 6.3]. The result then follows from (4). \square

Proposition 5.2. *Let X be a complete regular variety. Then*

$$\text{cdim}_p(X) \geq \begin{cases} (p-1) \cdot (n_p(X) - d_p(X)) & \text{if } k \text{ has characteristic } \neq p, \\ (p-1) \cdot \min(p, n_p(X) - d_p(X)) & \text{if } k \text{ has characteristic } = p. \end{cases}$$

Proof. Let $Z \subset X$ be a closed subvariety admitting a correspondence $X \rightsquigarrow Z$ of multiplicity prime to p , and such that $\dim Z = \text{cdim}_p(X)$. By Proposition 4.7 we have $d_p(Z) \leq d_p(X)$. Since there is a morphism $Z \rightarrow X$, we have $n_p(X) \leq n_p(Z)$. We conclude by applying Theorem 5.1 to the complete variety Z . \square

6. EXAMPLES

In view of section 5, it may seem desirable to find conditions on a variety X that give upper bounds for $d_p(X)$.

Proposition 6.1. *Let X be a complete smooth variety. Assume that there is a field extension l/k , and a complete smooth l -variety Y such that $X_l \times Y$ is a rational l -variety. Then $d_X = 1$.*

Proof. It will be sufficient to prove that $\chi(X, \mathcal{O}_X) = \pm 1$. While doing so, we may extend scalars, and thus assume that $k = l$. The variety $Z = X \times Y$ is then rational. Since the coherent cohomology groups of the structure sheaf are birational invariants of a complete smooth variety [CR11, Theorem 3.2.8], so is its Euler characteristic. The structure sheaf of the projective space has Euler characteristic equal to 1, hence $\chi(Z, \mathcal{O}_Z) = 1$. Since $\chi(Z, \mathcal{O}_Z) = \chi(X, \mathcal{O}_X) \cdot \chi(Y, \mathcal{O}_Y)$ [Ful98, Example 15.2.12], we are done. \square

Corollary 6.2. *Let X be a complete, smooth, geometrically rational variety. Then $d_X = 1$.*

Example 6.3 (Projective homogeneous varieties). Let X be a complete, smooth, geometrically connected variety, which is homogeneous under a semi-simple linear algebraic group. Then X is geometrically rational. Thus by Corollary 6.2, we have $d_X = 1$.

Proposition 6.4. *Assume that k has characteristic zero. Let X be a complete, smooth, geometrically rationally connected variety. Then $d_X = 1$.*

Proof. We proceed as in the proof of Proposition 6.1, and assume that X is rationally connected. Then the groups $H^i(X, \mathcal{O}_X)$ vanish for $i > 0$ by [Deb01, Corollary 4.18, a)], and therefore $\chi(X, \mathcal{O}_X) = 1$. \square

Remark 6.5 (Decomposition of the diagonal). The statement of Proposition 6.4 is more generally true (still in characteristic zero) under the assumption that the diagonal decomposes. This is the case when, for \bar{k} is an algebraic closure of k , the variety $\bar{X} = X_{\bar{k}}$ is complete and smooth, and satisfies $\text{CH}_0(\bar{X}_{\Omega}) = \mathbb{Z}$, where Ω is an algebraic closure of the function field of \bar{X} (see e.g. the introduction of [Esn03]).

Example 6.6 (Complete intersections). Let H be a complete intersection of hypersurfaces in \mathbb{P}^n of degrees $\delta_1, \dots, \delta_m$, with $\delta_1 + \dots + \delta_m \leq n$. When H is smooth, it is Fano, hence geometrically rationally chain connected. If, in addition, k has characteristic zero, the variety H is geometrically rationally connected, and Proposition 6.4 shows that $\chi(H, \mathcal{O}_H) = 1$. Alternatively, a direct computation shows that this is true in general (in any characteristic, for possibly singular H). This can be used to produce other sufficient conditions on a complete variety X for the equality $d_X = 1$, namely:

- X becomes isomorphic to such an H after extension of the base field,
- or X is smooth and becomes birational to such a smooth H after extension of the base field.

Example 6.7 (Hypersurfaces). Let H be a regular hypersurface of degree p and dimension $\geq p - 1$. By Example 6.6, we have $d_p(H) = 0$. Assume

that H has no closed point of degree prime to p . Then by Proposition 5.2, we have $\text{cdim}_p(H) \geq p - 1$.

In case $\dim H = p - 1$, this bound is optimal, and H is p -incompressible (see also [Mer03, § 7.3] and [Zai10, Example 6.4]). A more general statement was proved in [Hau12a, Proposition 10.1].

When $p = 2$, we have $\text{cdim}_2(H) \geq 1$. This bound is sharp when $\dim H = 2$, as can be seen by taking for H an anisotropic smooth projective Pfister quadric surface. In general this is far from being sharp: in characteristic not two, it is known that $\text{cdim}_2(H) = 2^n - 1$, where n is such that $2^n - 1 \leq \dim H < 2^{n+1} - 1$ (see [Mer03, § 7]).

Example 6.8 (A separation theorem). Let X and Y be complete varieties, with X regular. Assume that the degree of any closed point of Y is divisible by p , and that

$$\dim Y < p - 1 \leq \dim X.$$

Assume that X is a hypersurface of degree p , resp. a geometrically rationally connected variety over a field of characteristic zero. Then the degree of any closed point of $Y_{k(X)}$ is divisible by p . (Indeed, assuming the contrary, there would be a correspondence $X \rightsquigarrow Y$ of degree prime to p . Since by Example 6.6, resp. Proposition 6.4, we have $d_p(X) = 0$, Proposition 4.7 would imply that $d_p(Y) = 0$, which in turns implies that $n_p(Y) = 0$ by Theorem 5.1.)

Proposition 6.9. *Let $f: Y \dashrightarrow X$ be a rational map between regular complete varieties. Let F be the generic fiber of f , considered as a $k(X)$ -variety. Then*

- (i) *If $d_p(F) = 0$ then $d_p(X) = d_p(Y)$.*
- (ii) *We have $n_p(Y) \leq n_p(X) + n_p(F)$.*

Proof. Let us prove (i). The map f induces a correspondence $Y \xleftarrow{g} Z \xrightarrow{h} X$ with g birational. By Lemma 4.1 we have $d_Y = d_Z$.

We now prove the equality $d_p(X) = d_p(Z)$. We have a cartesian square

$$\begin{array}{ccc} F & \xrightarrow{\rho} & Z \\ \downarrow l & & \downarrow h \\ k(X) & \xrightarrow{\eta} & X \end{array}$$

Since $d_p(F) = 0$, there is $v \in K'_0(F) \otimes \mathbb{Z}_{(p)}$ such that

$$l_*v = 1 \in \mathbb{Z}_{(p)} = K'_0(k(X)) \otimes \mathbb{Z}_{(p)}.$$

Since $\rho^*: K'_0(Z) \rightarrow K'_0(F)$ is surjective (by the localisation sequence), we can find $u \in K'_0(Z) \otimes \mathbb{Z}_{(p)}$ such that $\rho^*u = v$. Then we have

$$\eta^* \circ h_*u = l_* \circ \rho^*u = l_*v = 1 = \eta^*[\mathcal{O}_X].$$

It follows that the element $h_*u - [\mathcal{O}_X]$ is in the kernel of η^* , hence is nilpotent; therefore h_*u is invertible in $K'_0(X) \otimes \mathbb{Z}_{(p)}$. We can now conclude that $d_p(X) = d_p(Z)$, as in Lemma 4.1.

To prove (ii), take a closed point of F of degree d , such that $v_p(d) = n_p(F)$. Its closure in Z gives a correspondence $X \rightsquigarrow Y$ of multiplicity d , and we can conclude using Proposition 4.5 (for more details, note that $F \subset Y_{k(X)}$, and see the proof of Proposition 7.5 below). \square

Example 6.10. Let p be an odd prime. Let H be a regular hypersurface of degree p and dimension $p-1$, with $n_p(H) = 1$. Let X be a regular complete variety admitting a rational map $X \dashrightarrow H$, whose generic fiber F is also a hypersurface of degree p and dimension $p-1$. Then $d_p(F) = d_p(H) = 0$ by Example 6.6, and therefore $d_p(X) = 0$ and $n_p(X) \leq 2$ by Proposition 6.9.

If $n_p(X) = 2$, then $\text{cdim}_p(X) = \dim X = 2(p-1)$ by Proposition 5.2.

If $n_p(X) = 1$, then $\text{cdim}_p(X) \geq p-1$. This bound is sharp: if $n_p(F) = 0$, then X and H are p -equivalent, hence have the same canonical p -dimension; but $\text{cdim}_p(H) = \dim H = p-1$ by Proposition 5.2.

7. INVARIANTS OF FUNCTION FIELDS

Definition 7.1. Let K/k be a finitely generated field extension. Let M be a regular complete variety such that $k(M)$ contains K as a k -subalgebra with $[k(M) : K]$ finite and prime to p . We define $d_p(K/k) := d_p(M)$ and $n_p(K/k) := n_p(M)$. By Lemma 7.4 below, and Corollary 4.3, these integers do not depend on the choice of M . If there is no such M , we set $d_p(K/k) = n_p(K/k) = \infty$ (by Gabber's theorem, this may only happen in characteristic p).

An alternative definition of $n_p(K/k)$ can be found in [Mer03, Remark 7.7]. Note that, when X is a complete variety, we have

$$d_p(X) \leq d_p(k(X)/k) \quad \text{and} \quad n_p(X) \leq n_p(k(X)/k),$$

with equalities when X is regular.

Remark 7.2. This process can be used more generally to define invariants $H_{K/k}(k) \otimes \mathbb{Z}_{(p)}$, when the pair (R, H) satisfies Conditions 3.1.

Remark 7.3 (Geometrically unirational field extensions). When k has characteristic zero, it is possible to have some control on $d_p(K/k)$ without explicitly introducing a variety M as in Definition 7.1. Namely assume that

there is a field extension l/k such that the ring $K \otimes_k l$ is contained in a purely transcendental field extension of l . Then $d_p(K/k) = 0$ for any p . (Indeed by Hironaka's resolution of singularities [Hir64], there is a smooth complete variety M with function field K . Then M is geometrically unirational, hence geometrically rationally connected. Thus by Proposition 6.4, we have $d_M = 1$.)

Lemma 7.4. *Let $X_1 \rightarrow X \leftarrow X_2$ be a diagram of varieties and proper morphisms, generically finite of degree prime to p . Then X_1 is p -equivalent to X_2 .*

Proof. Consider the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g_1} & X_2 \\ g_2 \downarrow & & \downarrow f_2 \\ X_1 & \xrightarrow{f_1} & X \end{array}$$

We claim that X' has an irreducible component which is generically finite of degree prime to p over X_1 ; this component then gives a correspondence $X_1 \rightsquigarrow X_2$ of multiplicity prime to p , and the lemma follows by symmetry.

Since f_i is generically finite and dominant, and X_i is integral, the generic fiber of f_i is the spectrum of $k(X_i)$ (for $i = 1, 2$). The ring $B = k(X_1) \otimes_{k(X)} k(X_2)$ is artinian, and the generic fiber of g_2 is $\text{Spec } B$. Write $B = B_1 \times \cdots \times B_n$, with B_1, \dots, B_n artinian local rings. The $k(X_1)$ -vector space B has dimension equal to $[k(X_2) : k(X)]$, an integer prime to p by hypothesis. Since $\dim B = \dim B_1 + \cdots + \dim B_n$, we can find an index i such that B_i has dimension prime to p over $k(X_1)$. The closure of $\text{Spec } B_i$ in X' then gives the required irreducible component. \square

Proposition 7.5. *Let K/k be a finitely generated field extension. Let X be a complete variety. Then*

$$n_p(X) \leq n_p(X_K) + n_p(K/k).$$

Proof. Let M be a complete regular variety such that $[k(M) : K]$ is finite and prime to p (if there is no such M , then $n_p(K/k) = \infty$, making the statement empty). Write $E = k(M)$. Then by a transfer argument

$$(5) \quad n_p(X_E) \leq n_p(X_K).$$

Let L/E be a finite field extension such that $X_E(L) \neq \emptyset$ and

$$(6) \quad v_p[L : E] = n_p(X_E).$$

Then the closure of the image of the map induced by an L -point of X_E

$$\mathrm{Spec} L \hookrightarrow X_E = (\mathrm{Spec} E) \times_k X \rightarrow M \times_k X$$

gives a correspondence $M \rightsquigarrow X$, whose multiplicity is $[L : E]$. By Proposition 4.5, we have

$$(7) \quad n_p(X) \leq v_p[L : E] + n_p(M).$$

The proposition then follows from (5), (6) and (7). \square

Proposition 7.6. *Let K/k be a finitely generated field extension. Then*

$$n_p(K/k) \leq d_p(K/k) + \left\lceil \frac{\mathrm{tr. deg}(K/k)}{p-1} \right\rceil.$$

Proof. We may assume that there is a complete regular variety M with $[k(M) : K]$ finite and prime to p (otherwise the statement is empty), and use Theorem 5.1. \square

Corollary 7.7. *Let K/k be a field extension of transcendence degree $< p-1$, with $d_p(K/k) = 0$. Then $n_p(K/k) = 0$.*

Corollary 7.8. *Let K/k be a field extension of transcendence degree $< p-1$. Assume that $d_p(K/k) = 0$ (see in particular Remark 7.3). Then*

- (i) *The relative Brauer group $\ker(\mathrm{Br}(k) \rightarrow \mathrm{Br}(K))$ has no p -primary torsion.*
- (ii) *Let α be a pure symbol in $K_*^M(k)/p$. Assume that k has characteristic zero. If $\alpha_K = 0$ then $\alpha = 0$.*

Proof. We have $n_p(K/k) = 0$ by Corollary 7.7. To prove (i), let A be central simple k -algebra of p -primary exponent. Take for X the Severi-Brauer variety of A . Then for any field extension l/k , the class of $A \otimes_k l$ vanishes in the Brauer group of l if and only if $n_p(X_l) = 0$. By Proposition 7.5, we have $n_p(X_K) = n_p(X)$, and (i) follows.

To prove (ii), one can take for X a complete generic p -splitting variety [SJ06] for α , and argue as above. \square

Example 7.9. Let D be a non-trivial p -primary central division k -algebra, and K/k a splitting field extension for D . Assume that K is the function field of a complete, smooth, geometrically rational variety (or more generally that $d_p(K/k) = 0$). Corollary 7.8, (i) says that $\mathrm{tr. deg}(K/k) \geq p-1$. One may ask whether we always have

$$\mathrm{tr. deg}(K/k) \geq \mathrm{ind}(D) - 1.$$

In other words, has the Severi-Brauer variety of D the smallest possible dimension among the complete, smooth, geometrically rational varieties whose function field splits D ?

Acknowledgements. I would like to thank Nikita Karpenko for his comments on this paper.

REFERENCES

- [AKMW02] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk. Torification and factorization of birational maps. *J. Amer. Math. Soc.*, 15(3):531–572 (electronic), 2002.
- [CR11] Andre Chatzistamatiou and Kay Rülling. Higher direct images of the structure sheaf in positive characteristic. *Algebra Number Theory*, 5(6):693–775, 2011.
- [Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.
- [Esn03] Hélène Esnault. Varieties over a finite field with trivial Chow group of 0-cycles have a rational point. *Invent. Math.*, 151(1):187–191, 2003.
- [Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, second edition, 1998.
- [Gil81] Henri Gillet. Riemann-Roch theorems for higher algebraic K -theory. *Adv. in Math.*, 40(3):203–289, 1981.
- [Hau] Olivier Haulion. Degree formula for the Euler characteristic. *Proc. Amer. Math. Soc.*, to appear. [arXiv:1103.4076](#).
- [Hau12a] Olivier Haulion. Integrality of the Chern character in small codimension. *Adv. Math.*, 231(2):855–878, 2012.
- [Hau12b] Olivier Haulion. Reduced Steenrod operations and resolution of singularities. *J. K-Theory*, 9(2):269–290, 2012.
- [Hir64] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)*, 79:109–203, 205–326, 1964.
- [ILO] Luc Illusie, Yves Laszlo, and Fabrice Orgogozo. *Travaux de Gabber sur l’uniformisation locale et la cohomologie étale des schémas quasi-excellents. Séminaire à l’Ecole polytechnique 2006–2008*. Avec la collaboration de Frédéric Déglise, Alban Moreau, Vincent Pilloni, Michel Raynaud, Joël Riou, Benoît Stroh et Michael Temkin. [arXiv:1207.3648](#).
- [Kar] Nikita A. Karpenko. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math.*, to appear.
- [KM] Nikita A. Karpenko and Alexander S. Merkurjev. On standard norm varieties. *Ann. Sci. École Norm. Sup.*, to appear.
- [KM06] Nikita A. Karpenko and Alexander S. Merkurjev. Canonical p -dimension of algebraic groups. *Adv. Math.*, 205(2):410–433, 2006.
- [LM07] Marc Levine and Fabien Morel. *Algebraic cobordism*. Springer Monographs in Mathematics. Springer, Berlin, 2007.
- [Mer03] Alexander Merkurjev. Steenrod operations and degree formulas. *J. Reine Angew. Math.*, 565:13–26, 2003.
- [Qui73] Daniel Quillen. Higher algebraic K -theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

- [Ros96] Markus Rost. Chow groups with coefficients. *Doc. Math.*, 1:No. 16, 319–393 (electronic), 1996.
- [sga71] *Théorie des intersections et théorème de Riemann-Roch*. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.
- [SJ06] Andrei Suslin and Seva Joukhovitski. Norm varieties. *J. Pure Appl. Algebra*, 206(1-2):245–276, 2006.
- [Wło03] Jarosław Włodarczyk. Toroidal varieties and the weak factorization theorem. *Invent. Math.*, 154(2):223–331, 2003.
- [Zai10] Kirill Zainoulline. Degree formula for connective K-theory. *Invent. Math.*, 179(3):507–522, 2010.

E-mail address: olivier.haution at gmail.com

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, UNIVERSITY PARK, NOTTINGHAM, NG7 2RD, UNITED KINGDOM